Translational tilings of the plane by a polygonal set

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I. Motivation

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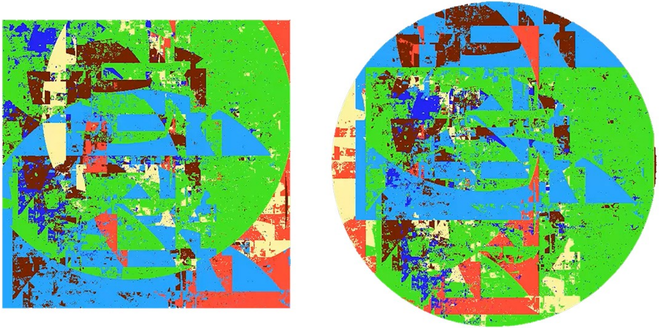

Tarski's circle squaring problem with measurable pieces.

Grabowski–Mathe–Pikhurko '16, Marks–Unger '16,
Mathe–Noel–Pikhurko '22

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Another perspective: Let $F \subseteq \mathbb{Z}$ be finite set that admits a tiling complement, is there an efficient local distributed algorithm that would produce $A \subseteq \mathbb{Z}$ such that $F \oplus A = \mathbb{Z}$.

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- *(Mycielski, Wagon) Is 3-divisibility of $\mathbb{S}^2$ possible with measurable pieces?*

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for every integer t coprime to q.

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Corollary (Bhattacharya '20)

Periodic tiling conjecture holds in $\mathbb{Z}^2$.

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Then $T \oplus A = \mathbb{Z}^m$ and the standard dilation lemma applies. $\square$

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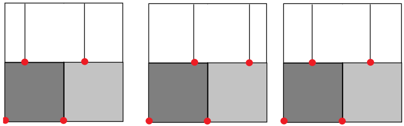

Figure 1: Red dots form the set of dilated translates T, T^5 and T^9.

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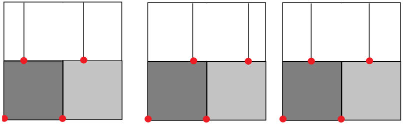

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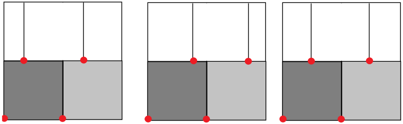

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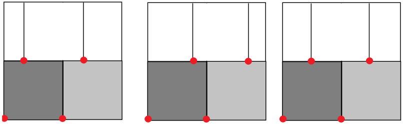

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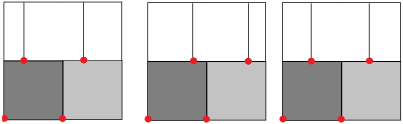

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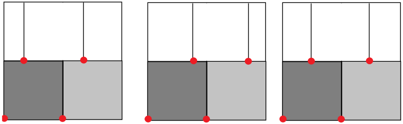

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- One gets rationality results in higher dimensions as well.

Application 4.

Joint work with de Dios Pont, Greenfeld and Madrid.

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- The same is true if we only tile a periodic set $E = \Omega \oplus T$.

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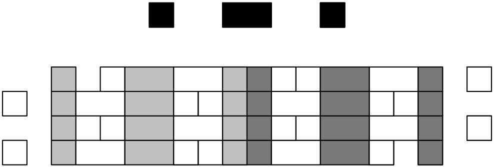

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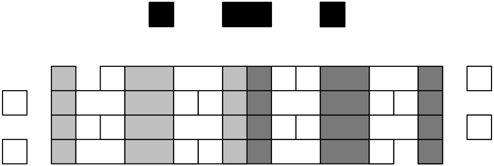

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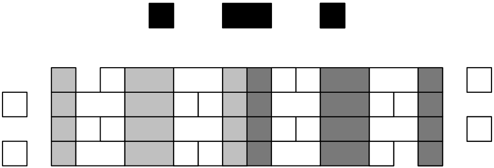

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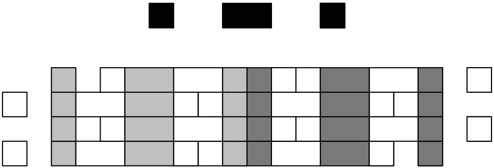

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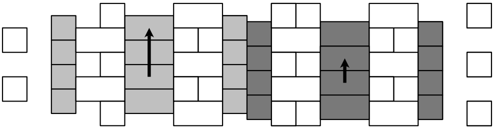

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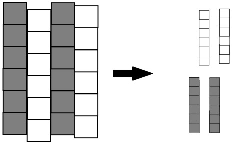

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- *T_c is a union of columns, that is, those $S \subseteq T$ such that $\Omega \oplus S$ is $\mathbb{R}(0,1)$-invariant and of the form $T \cap (\{x\} \times \mathbb{R})$ for some $x \in \mathbb{R}$.*

Thank you!